

Charge Oscillations in Superconducting Nanodevices Coupled to External Environments

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Abstract

Charge oscillations in certain nanodevices, more specifically the so-called Superconducting Cooper Pair Boxes (SCB), are usually interpreted as an effect of macroscopic quantum coherence; an alternative explanation is however possible in terms of the Gross-Pitaewski equation for the classical order parameter. These two explanations are based on different quantum states assigned to the SCB, occupation number states in the first case, coherent-like states in the second one. We show that, when the SCB is weakly coupled to an external source of noise and dissipation, occupation number states are much more unstable than coherent ones.

1. Among the various experimental implementations of qubits, the ones based on superconducting devices have recently attracted much interest. In particular, in a series of measures [1, 2] it has been shown that the so-called Superconducting Cooper Pair Boxes (SCB) [see section 2 below for a detailed definition] allow external control of charge oscillations. Because of the large number of Cooper pairs involved, these oscillations have been interpreted as a manifestation of macroscopic quantum coherence as theoretically explained by the so-called *quantum phase model* [3, 4] which essentially describes a quantized non-linear oscillator.

A different explanation of the experimental results is however possible in terms of the solution of a classical Gross-Pitaewski equation [5]. In this approach, the charge oscillations naturally emerge from the time-behaviour of the SCB order parameter on the basis of a mean-field treatment. While the quantum phase model describes the system in terms of occupation number states, the ones appearing in the mean-field formulation are close to coherent states.

Though the predicted frequencies of oscillations in the two models are different, the actual accuracy of the experimental data does not allow to discriminate between them. Nevertheless, as shown in the following, the two models behave quite differently in presence of noise induced by a weakly coupled external environment. In such a case, the SCB need to be treated as open quantum systems [6, 7, 8, 9]; as a consequence, their dynamics is no longer unitary, but it is described by a quantum dynamical semigroup, that takes into account effects of dissipation and decoherence induced by the environment.

In this respect, a physically important issue is whether the system is stable against the presence of noise. It occurs that the two models both predict relaxation, but with quite different decay properties. Taking into account the large number of Cooper pairs involved in the actual experimental implementations of SCB, the decay times of occupation number states turn out to be so large that charge oscillations would hardly be visible if based on the quantum phase model; on the other hand, they would be much less affected by the presence of noise when described in terms of coherent-like states.

This result suggests a possible experimental setting able to distinguish between the two models of the SCB charge oscillations. By experimentally inducing a stochastic perturbation of the Josephson tunneling, the charge oscillations would be immediately suppressed if due to macroscopic quantum coherence, while practically unaffected otherwise.

2. A SCB consists of a small superconductive island, labeled by “1” in the following, connected through a Josephson junction to a much larger superconductive island, labeled by “2”. The islands are inserted in a circuit with a control gate voltage V_g coupled to them via a gate capacitor C_g : in this way one can control the charge on the SCB; further, the total number, N , of Cooper pairs on the two islands is fixed.

A standard, widely used treatment to model the behaviour of the SCB is based on a non-linear oscillator classical Hamiltonian

$$H = 4E_C(n - n_g)^2 - E_J \cos \theta , \quad (1)$$

where E_C is the charging energy of the SCB, E_J the Josephson coupling energy, n the number of Cooper pairs in island 1 in excess with respect to a reference state (with a macroscopic average number of Cooper pairs \bar{n}_1), θ is the phase of the superconducting order parameter of the SCB, while $n_g = C_g V_g / (2e)$ accounts for the charging effect due to the control gate voltage [3, 4].

As mentioned in the Introduction, the above Hamiltonian is then quantized by considering the variable θ and n as conjugate phase and number operators satisfying canonical commutation relations. In the occupation number representation, the quantized Hamiltonian takes the form

$$\hat{H} = 4E_C \sum_{n=0}^N (n - n_g)^2 |n\rangle \langle n| - E_J \sum_{n=0}^N \left(|n\rangle \langle n+1| + |n+1\rangle \langle n| \right) . \quad (2)$$

In the so called charge qubit regime, the potential barrier between the two islands is small so that $E_C \gg E_J$. By adjusting the control gate voltage so that $n_g = 1/2$, one enters a

particular situation where only the states $|0\rangle$ and $|1\rangle$ play a role and are strongly coupled by the Josephson junction. In such a case, neglecting constant terms, the Hamiltonian (2) can be approximated by a two-level Hamiltonian,

$$\hat{H}_{\text{qubit}} = -\frac{4E_C(1-2n_g)}{2}\sigma_z - \frac{E_J}{2}\sigma_x, \quad (3)$$

with $\sigma_{x,z}$ Pauli matrices, effectively modeling a qubit. As a consequence, given an initial superposition $|\varphi\rangle$ of the eigenstates $|0\rangle, |1\rangle$, of (3), the probability of being in the fundamental state at time t ,

$$p(t) := \left| \langle 0 | e^{-it\hat{H}_{\text{qubit}}} | \varphi \rangle \right|^2, \quad (4)$$

shows coherent oscillations with frequency E_J . These oscillations have been experimentally detected [1, 2] opening the possibility to perform qubit operations by controlling the applied gate voltage and by inserting external magnetic fields.

Observe that an equivalent, more transparent interpretation of this behaviour of a SCB can be derived within a purely quantum framework by starting from a microscopic description in terms of bosonic annihilation and creation operators a_1, a_1^\dagger and a_2, a_2^\dagger of Cooper pairs in island 1 and 2, with fixed total number $N = a_1^\dagger a_1 + a_2^\dagger a_2$. Indeed, in a suitable regime, the SCB dynamics can be effectively modelled by the Bose-Hubbard Hamiltonian [10, 11]

$$H_{2\text{-mode}} = E(a_1^\dagger a_1)^2 + U_1 a_1^\dagger a_1 + U_2 a_2^\dagger a_2 - K(a_1 a_2^\dagger + a_1^\dagger a_2), \quad (5)$$

where the quadratic term $E(a_1^\dagger a_1)^2$ accounts for Coulomb repulsion in the small island (the one in the much larger island 2 can be neglected); further, $U_i a_i^\dagger a_i$, $i = 1, 2$, are potential contributions, while the last one is a tunneling term.

By substituting the operators a_i with complex numbers $\sqrt{\bar{n}_i} e^{i\theta_i}$, after some rearrangements and discarding constant terms, one gets the following classical Hamiltonian

$$H = E \left(n_1 - \frac{U_2 - U_1}{2E} \right)^2 - 2K n_1 (N - n_1) \cos \theta, \quad (6)$$

where $\theta := \theta_1 - \theta_2$. Let $n_1 = n + \bar{n}_1$, with n the number of excess Cooper pairs relative to the macroscopic average occupation number \bar{n}_1 in island 1. In SCB experimental applications, n is either 0 or 1 and therefore $n \ll \bar{n}_1$, the average occupation number \bar{n}_1 being typically of the order of 10^8 . One can then approximate $K n_1 (N - n_1)$ by a constant term $K \bar{n}_1 (N - \bar{n}_1)$ and the Hamiltonian (1) is thus recovered by setting $E = 4E_C$, $n_g := (U_2 - U_1)/2E - \bar{n}_1$ and $E_J = K \bar{n}_1 (N - \bar{n}_1)$. The standard qubit interpretation sketched before naturally follows, whereby the σ_z eigenstates $|0\rangle, |1\rangle$ really represent states with a macroscopic number of Cooper pairs. They are indeed orthogonal Fock states, that differ by one Cooper pair in island “1”.

3. A different description of the system is possible by treating the Bose-Hubbard Hamiltonian in a mean-field approach; this is justified by the large number of Cooper pairs involved.

A generic single Cooper pair state is of the form

$$|\psi\rangle = (\psi_1 a_1^\dagger + \psi_2 a_2^\dagger)|\text{vac}\rangle , \quad (7)$$

where $|\text{vac}\rangle$ denotes the vacuum state. It is the coherent superposition of a Cooper pair sitting on island 1 and a Cooper pair sitting on island 2, with amplitudes $\psi_i \in \mathbb{C}$ such that $|\psi_1|^2 + |\psi_2|^2 = 1$. The condensed state of the SCB is thus appropriately described by the product state

$$|\Psi\rangle_N := \underbrace{|\psi\rangle \otimes \cdots \otimes |\psi\rangle}_{N \text{ times}} = \frac{1}{\sqrt{N!}} (a^\dagger(\psi))^N |\text{vac}\rangle \quad (8)$$

$$= \sum_{k=0}^N \sqrt{\binom{N}{k}} \psi_1^k \psi_2^{N-k} |1_{(k)} 2_{(N-k)}\rangle , \quad (9)$$

where

$$|1_{(k)} 2_{(N-k)}\rangle := \frac{(a_1^\dagger)^k (a_2^\dagger)^{N-k}}{\sqrt{k!(N-k)!}} |\text{vac}\rangle , \quad (10)$$

is the occupation number state containing k Cooper pairs in island 1 and $N - k$ in island 2. More specifically, $|\Psi\rangle_N$ embodies the fact that on islands 1 and 2 there is a condensate consisting of a macroscopic number $N \gg \bar{n}_1$ of pairs in a same state $|\psi\rangle$ as in (7). Its structure is quite different from the number states that are used in the qubit approach: as already remarked at the end of the previous Section, these correspond to the occupation number states as in (10) which contain roughly \bar{n}_1 pairs on island 1 and $N - \bar{n}_1$ on island 2. On the other hand, as shown in (9), $|\Psi\rangle_N$ is a coherent superposition of many occupation number states on the two islands.

One can easily show that the following relations hold

$$a_1 |\Psi\rangle_N = \sqrt{N} \psi_1 |\Psi\rangle_{N-1} , \quad (11)$$

$$a_2 |\Psi\rangle_N = \sqrt{N} \psi_2 |\Psi\rangle_{N-1} ; \quad (12)$$

$$n_1 := {}_N\langle \Psi | a_1^\dagger a_1 | \Psi \rangle_N = N |\psi_1|^2 , \quad (13)$$

$$n_2 := N - n_1 = {}_N\langle \Psi | a_2^\dagger a_2 | \Psi \rangle_N = N |\psi_2|^2 . \quad (14)$$

The dynamics of $|\Psi\rangle_N$ follows the standard Schrödinger equation governed by the Hamiltonian (5)

$$i \frac{d}{dt} |\Psi_t\rangle_N = H_{2\text{-mode}} |\Psi_t\rangle_N . \quad (15)$$

Using (11) and (13) one finds for the two sides of this equation

$$i \frac{d}{dt} |\Psi_t\rangle_N = i \sqrt{N} \left(\dot{\psi}_1(t) a_1^\dagger |\Psi_t\rangle_{N-1} + \dot{\psi}_2(t) a_2^\dagger |\Psi_t\rangle_{N-1} , \right) \quad (16)$$

while

$$\begin{aligned}
H_{2\text{-mode}}|\Psi_t\rangle_N = & U_1\sqrt{N}\psi_1(t) a_1^\dagger|\Psi_t\rangle_{N-1} + U_2\sqrt{N}\psi_2(t) a_2^\dagger|\Psi_t\rangle_{N-1} \\
& - K\sqrt{N}\psi_1(t) a_2^\dagger|\Psi_t\rangle_{N-1} - K\sqrt{N}\psi_2(t) a_1^\dagger|\Psi_t\rangle_{N-1} \\
& + E\sqrt{N}\psi_1(t) (a_1^\dagger a_1) a_1^\dagger|\Psi_t\rangle_{N-1} .
\end{aligned} \tag{17}$$

Substituting $a_1^\dagger a_1$ in the last line with its mean value in (13), the product state $|\Psi_t\rangle_N$ is a solution of the evolution equation (15) if the amplitudes $\psi_i(t)$ solve the Gross-Pitaevski equations for the two component order parameter (ψ_1, ψ_2)

$$i\dot{\psi}_1 = E|\psi_1|^2\psi_1 + U_1\psi_1 - K\psi_2 \tag{18}$$

$$i\dot{\psi}_2 = U_2\psi_2 - K\psi_1 . \tag{19}$$

By setting $\psi_{1,2} := \sqrt{n_{1,2}} e^{i\theta_{1,2}}$ and using the same definitions as after (6), in particular $n_1 := n + \bar{n}_1$ and $\theta := \theta_2 - \theta_1$, these two equations reduce to

$$\dot{n} = -E_J \sin \theta , \quad \dot{\theta} = \frac{E}{2} n \tag{20}$$

whence, for small θ , the excess charge number n oscillates with frequency $\sqrt{(E_J E)/2}$.

As a consequence, the mean-field approach also provides an explanation for the charge oscillations experimentally observed in SCB, though possibly with a different frequency with respect to the one predicted by the quantum phase model. However, the experimental setups used so far are unable to measure frequency oscillations with sufficient accuracy and therefore to discriminate between the two explanations.

4. From the previous sections, it follows that there are two possible approaches to describe charge oscillations in SCB, the quantum phase model which is based on a purely quantum description and the mean-field one which is semiclassical in nature. They are both physically consistent with the experimental data; although they differ in the prediction of the oscillation frequencies, this is hardly relevant from the actual experimental viewpoint. Instead, we shall now show that the two approaches give different relaxation patterns when the SCB is weakly coupled to an environment which acts as a source of noise and dissipation; this result may indeed have experimental relevance.

We describe the weak coupling of the SCB to an external environment by means of the total Hamiltonian

$$\begin{aligned}
H_T = & H_E + E(a_1^\dagger a_1)^2 + U_1 a_1^\dagger a_1 + U_2 a_2^\dagger a_2 \\
& - K(a_1 a_2^\dagger + a_1^\dagger a_2) + \lambda \left(a_1 a_2^\dagger \otimes B + a_1^\dagger a_2 \otimes B^\dagger \right) ,
\end{aligned} \tag{21}$$

where H_E is the Hamiltonian of the environment, $\lambda \ll 1$ is a small coupling constant and B is a suitable environment operator.

We shall assume the environment to be in an equilibrium state and the corresponding two-point functions to be of white noise type, $\langle B^\dagger(t)B \rangle_E \simeq \delta(t)$. Among others, two possible environments exhibit such behaviour in their correlation functions: a heat bath in which the SCB is immersed at a temperature very large with respect to the SCB energy scales or an external classical stochastic field. In particular, the heat bath could consist of non-condensed electrons in the SCB, whereas the stochastic perturbation can be practically implemented thorough a random variation of the Josephson tunneling term and therefore be externally controlled.

The delta-like environment correlation functions provide the setting for the so called *singular coupling limit* [12] that leads to a master equation for SCB density matrices ρ of Kossakowski-Lindblad form:

$$\partial_t \rho = -i[H_{2\text{-mode}} + H^{(2)}, \rho] + \mathbb{D}[\rho] \quad (22)$$

$$\begin{aligned} \mathbb{D}[\rho] = & \gamma \left(b\rho b^\dagger - \frac{1}{2}\{b^\dagger b, \rho\} \right) + \delta \left(b^\dagger \rho b - \frac{1}{2}\{bb^\dagger, \rho\} \right) \\ & + \beta \left(b\rho b - \frac{1}{2}\{b^2, \rho\} \right) + \beta^* \left(b^\dagger \rho b^\dagger - \frac{1}{2}\{(b^\dagger)^2, \rho\} \right), \end{aligned} \quad (23)$$

where $b := a_1 a_2^\dagger$ and

$$\gamma := \lambda^2 \int_{-\infty}^{+\infty} ds \langle B^\dagger(s)B \rangle_E, \quad (24)$$

$$\delta := \lambda^2 \int_{-\infty}^{+\infty} ds \langle B(s)B^\dagger \rangle_E, \quad (25)$$

$$\beta := \lambda^2 \int_{-\infty}^{+\infty} ds \langle B(s)B \rangle_E. \quad (26)$$

The extra Hamiltonian term $H^{(2)}$ is a bath induced correction and the coefficients $\beta \in \mathbb{C}$, $\gamma \geq 0$ and $\delta \geq 0$ satisfy the condition $\gamma\delta \geq |\beta|^2$ which ensures the complete positivity [6, 7, 8] of the one-parameter quantum dynamical semigroup generated by (22). It incorporates the dissipative effects on the dynamics of the SCB due to the presence of an external environment.

It is enough to limit our considerations to the study of the stability properties of any initial SCB pure state $|\phi\rangle\langle\phi|$. We shall thus focus on the decay constant at time $t = 0$, to which only the term $\mathbb{D}[\rho]$ in (22) contributes,

$$\begin{aligned} \Gamma_\phi := -\langle\phi|\mathbb{D}[|\phi\rangle\langle\phi|]|\phi\rangle = & \gamma \left(|\langle\phi|b|\phi\rangle|^2 - \langle\phi|b^\dagger b|\phi\rangle \right) + \delta \left(|\langle\phi|b|\phi\rangle|^2 - \langle\phi|bb^\dagger|\phi\rangle \right) \\ & + 2\Re \left(\beta \left((\langle\phi|b|\phi\rangle)^2 - \langle\phi|b^2|\phi\rangle \right) \right), \end{aligned} \quad (27)$$

where \Re stands for real part; Γ_ϕ measures the velocity with which any state $|\phi\rangle\langle\phi|$ initially departs from itself.

We shall show that the decay constant Γ_ϕ strongly depends on whether the initial state of the SCB is an occupation number (Fock) or a coherent-like state. We stress again that

the first choice underlies the widely accepted qubit interpretation of the charge oscillations, while the other is at the root of the mean-field approach.

• Quantum phase model

As already mentioned at the end of Section 2, the states that play a role in the qubit explanation of SCB oscillations are occupation number states. Thus, we shall consider as initial state $|\phi\rangle$ the eigenstate $|0\rangle$ of σ_z , which is of the form $|\phi\rangle = |1_{(n_1)}2_{(N-n_1)}\rangle$, with $n_1 \simeq \bar{n}_1 \simeq 10^8$. By observing that

$$b|1_{(n_1)}2_{(N-n_1)}\rangle = a_1 a_2^\dagger |1_{(n_1)}2_{(N-n_1)}\rangle = \sqrt{n_1(N-n_1+1)}|1_{(n_1-1)}2_{(N-n_1+1)}\rangle \quad (28)$$

$$b^\dagger|1_{(n_1)}2_{(N-n_1)}\rangle = a_1^\dagger a_2 |1_{(n_1)}2_{(N-n_1)}\rangle = \sqrt{(n_1+1)(N-n_1)}|1_{(n_1+1)}2_{(N-n_1-1)}\rangle \quad (29)$$

$$b^\dagger b|1_{(n_1)}2_{(N-n_1)}\rangle = n_1(N-n_1+1)|1_{(n_1)}2_{(N-n_1)}\rangle \quad (30)$$

$$bb^\dagger|1_{(n_1)}2_{(N-n_1)}\rangle = (n_1+1)(N-n_1)|1_{(n_1)}2_{(N-n_1)}\rangle \quad (31)$$

the decay constant is computed to be

$$\Gamma_{\text{qubit}} = \gamma n_1(N-n_1+1) + \delta(n_1+1)(N-n_1) . \quad (32)$$

• Mean-field model

The states involved in this approach are instead of the form $|\Psi\rangle_N$ given in (8). Using relations (28)–(31), one can show that

$${}_N\langle\Psi|b|\Psi\rangle_N = \sqrt{n_1(N-n_1)}e^{i\theta} \quad (33)$$

$${}_N\langle\Psi|b^\dagger b|\Psi\rangle_N = n_1\left(N-n_1+\frac{n_1}{N}\right) \quad (34)$$

$${}_N\langle\Psi|bb^\dagger|\Psi\rangle_N = (N-n_1)\left(1+n_1-\frac{n_1}{N}\right) , \quad (35)$$

so that $|\Psi\rangle_N$ indeed behaves like a coherent state for the operator b , with labeling parameter $\sqrt{n_1(N-n_1)} \exp(i\theta)$.

With the help of the previous relations, the decay constant in the mean-field model explicitly reads

$$\Gamma_{\text{mean-field}} = \frac{n_1^2}{N}\gamma + \delta(N-n_1)\left(1-\frac{n_1}{N}\right) - 2\Re\left(\beta e^{2i\theta}\right)\frac{n_1(N-n_1)}{N} . \quad (36)$$

In typical experimental conditions, the average number n_1 of Cooper pairs in island 1 is essentially the macroscopic occupation number $\bar{n}_1 \simeq 10^8$, while the total number N of Cooper pairs in both islands is such that $\bar{n}_1 \ll N$. Taking this into account and comparing the two decay constants in (32) and (36), one finds

$$\frac{\Gamma_{\text{qubit}}}{\Gamma_{\text{mean-field}}} \simeq \bar{n}_1 . \quad (37)$$

As a consequence, because of the large value of \bar{n}_1 , the two decay constants turn out to be very different in magnitude.

In presence of an external source of noise and dissipation, occupation number states in the quantum phase model seem to be much more unstable than the semiclassical, coherent like states in the mean-field approach. This gives the possibility of distinguishing the two descriptions. What looks experimentally implementable is a setup in which an externally controlled stochastic white noise is injected into the SCB device, for instance by suitably modifying the Josephson junction characteristic parameters and thus the potential barrier. By switching on such a stochastic perturbation, charge oscillations should be suppressed if due to macroscopic quantum coherence as described by the quantum phase model; instead, if they survive, that would be an indication of a semiclassical origin as modeled by the mean field approach.

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